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The Complexity of Explicit Definitions

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The Beth definability theorem cannot be refined to provide bounds on the complexity of an explicit definition effectively from the given implicit definition, even if the implicit definitions are assumed to be universal. There is no bound on the complexity of an explicit definition even if the implicit definition is assumed to consist of four universalized equations.

We begin with a careful statement of what we will call the (X, Y, Z)-definability theorem.

A relational type is a set X of constant, relation, and function symbols. An X-formula is a formula in the first-order predicate calculus with identity, whose only nonlogical symbols are in X. An X-structure consists of a nonempty domain D together with interpretations of the symbols in X on D. If $\mathfrak{A}$ is a structure and X is a relational type, then $\mathfrak{A}_X$ is the structure obtained by simply deleting, from $\mathfrak{A}$, the interpretations of the nonlogical symbols not in X. We write $T \models \psi$, or $\varphi \models \psi$, to indicate that every model of T, or φ, is a model of ψ.

The (X, Y)-definability theorem states the following. Let X, Y be disjoint finite relational types, $X \cup Y \subset Z$. Let ψ be a Z-formula. Suppose that for all Z-structures $\mathfrak{A}$, $\mathfrak{B} \models \psi$, if $\mathfrak{A}_X = \mathfrak{B}_X$ then $\mathfrak{A} = \mathfrak{B}$. Then for each constant symbol $c \in Y$, each n-ary relation symbol $R \in Y$, and each function symbol $F \in Y$, there are X-formulas $A_c(x_1)$, $B_R(x_1 ,..., x_n)$, and $C_F(x_1 ,..., x_n , x_{n+1})$ with only the free variables shown, such that $\psi \models (x_1 = c \leftrightarrow A_c(x_1))$ & $(R(x_1 ,..., x_n) \leftrightarrow B_R(x_1 ,..., x_n))$ & $(F(x_1 ,..., x_n) = x_{n+1} \leftrightarrow C_F(x_1 ,..., x_n , x_{n+1}))$.

We will also consider the following weak form of the (X, Y, Z)-definability theorem, which we will refer to as the weak (X, Y, Z)-

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definability theorem. Let X, Y be disjoint finite relational types, $X \cup Y \subset Z$. Let φ be an X-formula and ψ be a Z-formula. Suppose that for all X-structures $\mathfrak{A} \models \varphi$ there is a unique Z-structure $\mathfrak{B} \models \psi$ with $\mathfrak{B}_X = \mathfrak{A}$. Then for each constant symbol $c \in Y$, each n-ary relation symbol $R \in Y$, and each function symbol $F \in Y$, there are X-formulas $A_c(x_1)$, $B_R(x_1, \ldots, x_n)$, and $C_F(x_1, \ldots, x_n, x_{n+1})$ with only the free variables shown, such that $\{\varphi, \psi\} \models (x_1 = c \leftrightarrow A_c(x_1))$ & $(R(x_1, \ldots, x_n) \leftrightarrow B_R(x_1, \ldots, x_n))$ & $(F(x_1, \ldots, x_n) = x_{n+1} \leftrightarrow C_F(x_1, \ldots, x_n, x_{n+1}))$.

There is one further weakening that we shall consider. By the superweak (X, Y, Z)-definability theorem, we shall mean the weak (X, Y, Z)-definability theorem restricted to tautologous φ.

In the usual formulations of Beth's theorem, $Z = X \cup Y$. We will write (X, Y) instead of $(X, Y, X \cup Y)$.

The (X, Y, Z)-definability theorem (and hence the weak and superweak definability theorems) are partial effectively true, in the sense that there is a partial recursive function which takes gödel numbers of φ obeying the hypotheses of the theorem into gödel numbers of formulas obeying its conclusions. In [2] it is shown that there is no such total recursive function. Actually, what is shown there is somewhat stronger: There is none that takes such numbers into a bound on the number of symbols needed to write down formulas obeying the conclusion. In [1] it is shown that one cannot obtain, even recursively, a bound on the number of quantifier alterations needed. This paper is devoted to a number of improvements of that result in [1].

The $\prod_0$ formulas, and the $\sum_0$ formulas, are just the quantifier free formulas. The $\prod_{n+1}$ formulas are the $\prod_n$ and $\sum_n$ formulas together with the formulas of the form $(\forall y_1) \cdots (\forall y_k)(A)$, where A is a $\sum_n$ formula. The $\sum_{n+1}$ formulas are the $\prod_n$ and $\sum_n$ formulas together with the formulas of the form $(\exists y_1) \cdots (\exists y_k)(A)$, where A is a $\prod_n$ formula.

For every set T of formulas, let $\mathscr{L}(T)$ be the set of all nonlogical symbols appearing in T.

We say that $\mathfrak{A}$ is an ω-structure, or ω-model just in case the relational type of $\mathfrak{A}$ includes 0, $'$, the domain of $\mathfrak{A}$ is ω, 0 is interpreted as 0, and $'$ is interpreted as the sucessor function.

Let T_0 consist of the formulas $\sim(x' = 0)$, $x' = y' \to x = y$, $x + 0 = x$, $x + y' = (x + y)'$, $x \cdot 0 = 0$, $x \cdot y' = (x \cdot y) + x$, $P(0) = 0$, $P(x') = x$, $x \dot{-} 0 = 0$, $x \dot{-} y' = P(x \dot{-} y)$, $Py(0) = 0$, $Py(x') = 0' \dot{-} Py(x)$, $x = (\frac{1}{2}(x) + \frac{1}{2}(x)) + Py(x)$, $\langle x, y \rangle = \frac{1}{2}((x + y) \cdot (x + y) + (0''' \cdot x + y))$, $\langle (x)_0, (x)_1 \rangle = x$, $\langle x, y \rangle = \langle z, w \rangle \to (x = z \,\&\, y = w)$, $x \neq 0 \to P(x)' = x$, $(x \dot{-} y = 0 \,\&\, y \dot{-} x = 0) \to x = y$.

LEMMA 1. *Let $\mathscr{L}(T_0) \subset X$, and let T be any finite set of X-formulas such that $T \models T_0$, and T has an ω-model. Then there is a finite $Y \supset X$ and a finite set T^* of quantifier free Y-formulas such that*

(i) *$T^* \models T$, and T^* has an ω-model.*

(ii) *Every element of $Y - X$ is a function symbol.*

(iii) *For n-ary $F \in Y - X$, there is an X-formula $\varphi(x_1, \ldots, x_n, x_{n+1})$ such that $T^* \models F(x_1, \ldots, x_n) = x_{n+1} \leftrightarrow \varphi(x_1, \ldots, x_n, x_{n+1})$.*

Proof. Let S_0 be a set of prenex X-sentences such that $T \models S_0$, $S_0 \models T$. Suppose S_k has been defined. Construct S_{k+1} as follows. For each sentence in S_k of the form $(\forall y_1) \cdots (\forall y_n)(\exists y)(Qw_1) \cdots (Qw_r)(A(y_1, \ldots, y_n, z, w_1, \ldots, w_r))$, where $1 \leqslant n$, $0 \leqslant r$, A is quantifier free, introduce an n-ary function symbol F and put the sentences $(\forall y_1) \cdots (\forall y_n)(Qw_1) \cdots (Qw_r)(A(y_1, \ldots, y_n, F(y_1, \ldots, y_n), w_1, \ldots, q_r)$, $(\forall y_1) \cdots (\forall y_n)(-Qw_1) \cdots (-Qw_r)(A(y_1, \ldots, y_n, z, w_1, \ldots, w_r) \rightarrow F(y_1, \ldots, y_n) \dot{-} z = 0)$ in S_{k+1}. Here $-Q$ is given by $-\forall = \exists$, $-\exists = \forall$. Take $T^* = \bigcup_k S_k$ stripped of all quantifiers.

We will call a formula $\sum_n^0(\prod_n^0)$, $0 < n$, just in case it is a prenex $\mathscr{L}(T_0)$-formula with exactly n quantifiers, beginning with an existential (universal) quantifier.

LEMMA 2. *There is a finite set $T_1 \models T_0$ of formulas such that*

(a) *$V(Sb(x, y, z), y) = z$, $u \neq y \rightarrow V(Sb(x, y, z), u) = V(x, u)$, $Rk(D(x)) = P(Rk(x))$, $Rk(x) \neq 0 \rightarrow (E(x) \vee U(x))$, $E(x) \rightarrow {\sim}U(x)$, $(H(x, y) \;\&\; H(x, z)) \rightarrow y = z$, $x \dot{-} \bar{k} = 0 \leftrightarrow \bigvee_{i \leqslant k} x = \bar{\imath}$, are consequences of T_1.*

(b) *$Rk(\bar{n}) = \bar{m}$, $Q(\bar{n}) = \bar{\jmath}$, $D(\bar{n}) = \bar{p}$, $E(\bar{q})$, $U(\bar{r})$, $\mathrm{Tr}(\bar{s}, x) \leftrightarrow \varphi(V(x, 0), \ldots, V(x, \bar{k}))$, $H(\bar{e}, x) \leftrightarrow x = \bar{t}$, are consequences of T_1, provided n is the gödel number of a $\sum_m^0$ or $\prod_m^0$ $\mathscr{L}(T_0)$-formula ψ, ψ begins with either $(\exists x_j)$ or $(\forall x_j)$, p is the gödel number of the result of deleting the leading quantifier $(\exists x_j)$ or $(\forall x_j)$ from ψ, q is the gödel number of some $\prod_m^0$ $\mathscr{L}(T_0)$-formula, $\varphi(x_0, \ldots, x_k)$ is a quantifier free $\mathscr{L}(T_0)$-formula with all free variables shown and gödel number s, $\varphi_e(0) \uparrow$ in exactly t steps.*

(c) *There are $\mathscr{L}(T_0)$-formulas $A_0(x, y)$, $A_1(x, y)$, $A_2(x, y)$, $A_3(x)$, $A_4(x)$, $A_5(x, y)$, $A_6(x, y)$, $A_7(x, y, z)$, $A_8(x, y, z, w)$ such that $Rk(x) = y \leftrightarrow A_0(x, y)$, $Q(x) = y \leftrightarrow A_1(x, y)$, $D(x) = y \leftrightarrow A_2(x, y)$, $E(x) \leftrightarrow A_3(x)$, $U(x) \leftrightarrow A_4(x)$, $\mathrm{Tr}(x, y) \leftrightarrow A_5(x, y)$, $H(x, y) \leftrightarrow A_6(x, y)$, $V(x, y) = z \leftrightarrow A_7(x, y, z)$, $Sb(x, y, z) = w \leftrightarrow A_8(x, y, z, w)$, are in T_1.*

(d) $\mathscr{L}(T_1) = \mathscr{L}(T_0) \cup \{C, D, E, U, \mathrm{Tr}, H, V, S\}$.

(e) *T_1 has an ω-model.*

Proof. Choose T_1 to be a suitable fragment of Peano arithmetic, together with suitable explicit definitions of new symbols.

Now let T_2 be the $T_1{}^*$ given by Lemma 1.

LEMMA 3. *T_2 is a finite set of quantifier free formulas, $T_2 \models T_1$, $\mathscr{L}(T_1) \subset \mathscr{L}(T_2)$, and T_2 has an ω-model. Furthermore, every atomic $\mathscr{L}(T_2)$-formula is, provably in T_2, equivalent to a $\mathscr{L}(T_0)$-formula.*

Proof. Clear from Lemmas 1 and 2.

For each e we introduce the set of sentences $K(e)$ consisting of

(1) T_2.

(2) $H(\bar{e}, c)$.

(3) $\mathrm{Sat}(x, y) \rightarrow Rk(x) \dot{-} c = 0$.

(4) $Rk(x) = 0 \rightarrow (\mathrm{Sat}(x, y) \leftrightarrow \mathrm{Tr}(x, y))$.

(5) $(Rk(x) \neq 0 \;\&\; Rk(x) \dot{-} c = 0 \;\&\; E(x)) \rightarrow (\mathrm{Sat}(x, y) \leftrightarrow (\exists z)(\mathrm{Sat}(D(x), Sb(y, Q(x), z))))$.

(6) $(Rk(x) \neq 0 \;\&\; Rk(x) \dot{-} c = 0 \;\&\; U(x)) \rightarrow (\mathrm{Sat}(x, y) \leftrightarrow (\forall z)(\mathrm{Sat}(D(x), Sb(y, Q(x), z))))$.

LEMMA 4. *There is a number p such that the following is true. If $\varphi_e(0)\uparrow$ in exactly $k + p$ steps, then*

(i) *for $\mathscr{L}(T_2)$-structures $\mathfrak{A} \models T_2$, there is a unique $\mathscr{L}(T_2)$-structure $\mathfrak{B} \models K(e)$ with $\mathfrak{B}_{\mathscr{L}(T_2)} = \mathfrak{A}$;*

(ii) *for each $\sum_k$ $\mathscr{L}(T_2)$-formula $\varphi(x, y)$, we have $K(e) \models \sim(\forall x)(\forall y)(\mathrm{Sat}(x, y) \leftrightarrow \varphi(x, y))$;*

(iii) *$K(e)$ has an ω-model.*

Proof. Choose p to be such that the second part of Lemma 3 holds with $\sum_p$ $\mathscr{L}(T_0)$-formulas. Then use a diagonal argument for (ii), together with the fact that every $\sum_n$ $\mathscr{L}(T_0)$-formula is, provably in T_0, equivalent to a $\sum_n^0$ formula.

LEMMA 5. *For each e one can effectively associate a consistent quantifier free formula $T(e)$ such that*

(i) $\mathscr{L}(T_2) \subset \mathscr{L}(T(e))$, $\mathscr{L}(T(e)) = \mathscr{L}(T(n))$;

(ii) *if* $\varphi_e(0)\uparrow$ *in exactly* $k + p$ *steps, then for* $\mathscr{L}(T(e))$*-structures* $\mathfrak{A}, \mathfrak{B} \models T(e)$, *if* $\mathfrak{A}_{\mathscr{L}(T_2)} = \mathfrak{B}_{\mathscr{L}(T_2)}$ *then* $\mathfrak{A} = \mathfrak{B}$;

(iii) *there is an* $R \in \mathscr{L}(T(e)) - \mathscr{L}(T_2)$ *such that for all* $\sum_k \mathscr{L}(T_2)$*-formulas* $C(x, y)$, *we have* $T(e) \models \sim(\forall x)(\forall y)(R(x, y) \leftrightarrow C(x, y))$.

Proof. Take $T(e)$ to be the conjunction of the set $K(e)^*$ given by Lemma 1.

THEOREM 1. *There are disjoint finite relational types* X, Y, *and* $R \in Y$ *such that the following holds. For each recursive function* ρ *on* ω, *there is a consistent quantifier free* $X \cup Y$*-formula* ψ *such that*

(i) *for* $X \cup Y$*-structures* $\mathfrak{A}, \mathfrak{B} \models \psi$, *if* $\mathfrak{A}_X = \mathfrak{B}_X$ *then* $\mathfrak{A} = \mathfrak{B}$;

(ii) *for all* $\sum_{\rho(\#(\psi))}$ X*-formulas* $C(x, y)$, *we have* $\psi \models \sim(\forall x)(\forall y)(R(x, y) \leftrightarrow C(x, y))$.

Proof. Let ρ be given. Since the halting problem is unsolvable, let e, k be such that $\varphi_e(0)\uparrow$ in exactly $k + p$ steps, and $\rho(\#(T(e))) \leqslant k$. Now just apply Lemma 5, and take $\psi = T(e)$.

COROLLARY 1. *There are disjoint finite relational types* X, Y, *such that the following holds. There is no recursive function* ρ *on* ω, *such that for all quantifier free* X*-formulas* ψ *obeying the hypotheses of the* (X, Y)*-definability theorem, there are* $\sum_{\rho(\#(\psi))}$ *formulas obeying its conclusion.*

Proof. The consistency of ψ in Theorem 1 is needed to obtain this.

LEMMA 6. *For each* $K(e)$, *one can effectively find a semantically equivalent* $\sum_1$ $\mathscr{L}(K(e))$*-formula* $L(e)$.

Proof. Elementary quantifier manipulations.

THEOREM 2. *There are disjoint finite relational types* X, Y, $R \in Y$, *and a quantifier free* X*-formula* φ, *such that the following holds. For each recursive function* ρ *on* ω, *there is a consistent* $\sum_1$ $X \cup Y$*-formula* ψ *such that*

(i) *for* X*-structures* $\mathfrak{A} \models \varphi$, *there is a unique* $X \cup Y$*-structure* $\mathfrak{B} \models \psi$ *with* $\mathfrak{B}_X = \mathfrak{A}$;

(ii) *for all* $\sum_{\rho(\#(\psi))}$ X*-formulas* $C(x,y)$, *we have* $\{\varphi, \psi\} \models \sim(\forall x)(\forall y)(R(x, y) \leftrightarrow C(x, y))$.

Proof. As in the proof of Theorem 1, using $L(e)$, and using & (T_2) for φ.

COROLLARY 2. *There are disjoint finite relational types X, Y, and a quantifier-free X-formula φ, such that the following holds. There is no recursive function ρ on ω, such that for all $\sum_1$ $X \cup Y$-formulas ψ obeying the hypotheses of the weak (X, Y)-definability theorem, there are $\sum_{\rho(\#(\psi))}$ formulas obeying its conclusion.*

LEMMA 7. *For each $K(e)$, one can effectively find a $\sum_1$ $\mathscr{L}(K(e))$-formula $M(e)$, such that $T_2 + M(e)$ is semantically equivalent to $K(e)$, and for $\mathscr{L}(T_2)$-structures $\mathfrak{A} \not\models T_2$, there is a unique $\mathscr{L}(M(e))$-structure $\mathscr{B} \models M(e)$ with $\mathscr{B}_{\mathscr{L}(T_2)} = \mathfrak{A}$.*

Proof. Another quantifier manipulation.

LEMMA 8. *If $\varphi_e(0)\uparrow$ in exactly $k + p$ steps, then*

(i) *for $\mathscr{L}(T_2)$-structures $\mathfrak{A}$, there is a unique $\mathscr{L}(M(e))$-structure $\mathscr{B} \models M(e)$;*

(ii) *for no $\sum_k$ $\mathscr{L}(T_2)$-formula $\varphi(x, y)$, do we have $M(e) \models \mathrm{Sat}(x, y) \leftrightarrow \varphi(x, y)$.*

Proof. From Lemmas 4 and 7.

THEOREM 3. *There are disjoint finite relational types X, Y, such that the following holds. There is no recursive function ρ on ω, such that for all $\sum_1$ $X \cup Y$-formulas ψ obeying the hypotheses of the superweak (X, Y)-definability theorem, there are $\sum_{\rho(\#(\psi))}$ formulas obeying its conclusion.*

COROLLARY 3. *There are disjoint finite relational types X, Y, $X \cup Y \subset Z$, and a quantifier free X-formula φ, such that the following holds. There is no recursive function ρ on ω, such that for all quantifier free Z-formulas ψ obeying the hypotheses of the superweak (X, Y, Z)-definability theorem, there are $\sum_{\rho(\#(\psi))}$ formulas obeying its conclusion.*

We do not know whether Corollary 2 holds with $\sum_1$ replaced by "quantifier free." The strongest result that we have is that there is a quantifier free X-formula φ and a quantifier free $X \cup Y$-formula ψ, obeying the hypotheses of the weak (X, Y)-definability theorem, such that no $\sum_2$ or $\prod_2$ formula obeys its conclusion. We presently give a proof of this.

Let K_0 consist of the formulas $\sim 1 = 0$, $\sim x < x$, $x < y \vee y < x \vee x = y$, $(x < y \,\&\, y < z) \rightarrow x < z$, $\langle (x)_0, (x)_1 \rangle = x$, $\langle x, y \rangle = \langle z, w \rangle \rightarrow (x = z \,\&\, y = w)$, $(\sim R(x, y, z) \,\&\, \sim R(x, w, z)) \rightarrow y = w$, $A(x, z) \rightarrow (a < (x)_0 \rightarrow (F(a, x, z) \leqslant (x)_1 \,\&\, \sim R(a, (x)_1, z)))$, $\sim A(x, z) \rightarrow (G(x, z) < (x)_0 \,\&\, (y \leqslant (x)_1 \rightarrow R(G(x, z), y, z)))$, $B(x, z) \rightarrow (H(x, z) < (x)_0 \,\&\, \sim(R(H(x, z), (x)_1, z))$, $\sim B(x, z) \rightarrow (a < (x)_0 \rightarrow R(a, (x)_1, z))$. Here $x \leqslant y$ abbreviates $x < y \vee x = y$.

LEMMA 9. *The following are provable in* K_0 : $A(x, z) \leftrightarrow (\forall a < (x)_0)(\exists y \leqslant (x)_1)(\sim R(a, y, z))$, $B(x, z) \leftrightarrow (\exists a < (x)_0)(\sim R(a, (x)_1, z))$.

Proof. F, G, H were introduced as Skolem functions for this purpose.

LEMMA 10. $K_0 \models (\exists x)((\forall y)(R((x)_0, y, z)) \,\&\, A(x, z) \,\&\, B(x, z)) \leftrightarrow (\exists !\, x)((\forall y)(R((x)_0, y, z)) \,\&\, A(x, z) \,\&\, B(x, z))$.

Now let K_1 consist of K_0 together with the formulas $L(z) = 0 \rightarrow (R((I(z))_0, y, z) \,\&\, A(I(z), z) \,\&\, B(I(z, z))$, $L(z) \neq 0 \rightarrow I(z) = 0$, $(\sim C(z) \,\&\, A(x, z) \,\&\, B(x, z)) \rightarrow \sim R((x)_0, J(x, z), z)$, $(L(z) \neq 0 \,\&\, \sim(A(x, z) \,\&\, B(x, z))) \rightarrow J(x, z) = 0$, $L(z) = 0 \rightarrow J(x, z) = 0$, $L(z) = 0 \vee L(z) = 1$.

LEMMA 11. $K_1 \models L(z) = 0 \leftrightarrow (\exists x)(\forall y)(R((x)_0, y, z)) \,\&\, A(x, z) \,\&\, B(x, z))$. *For* $\mathscr{L}(K_0)$*-structures* $\mathfrak{A} \models K_0$, *there is a unique* $\mathscr{L}(K_1)$*-structure* $\mathscr{B} \models K_1$ *with* $B_{\mathscr{L}(K_0)} = \mathfrak{A}$.

Proof. It is clear by Lemma 10 that I, J are uniquely determined from L. Since I, J are introduced as Skolem functions, L is determined as in the first half of the lemma.

LEMMA 12. *There is an* $\mathscr{L}(K_0)$*-structure* $\mathfrak{A}$ *with domain* ω, *in which* 0, 1 *are interpreted as* 0, 1, $<$ as $<$, $\langle\ \rangle$, $(\ )_0$, $(\ )_1$, F, G, H, *as certain recursive functions*, A, B, *as certain recursive relations, and* R *as a certain recursive relation in order that* $\{z\colon (\exists x)(\forall y)(R(x, y, z))\}$ *be a complete* $\sum_2^0$ *set of integers.*

LEMMA 13. $\mathfrak{A} \models L(z) = 0 \leftrightarrow (\exists x)(\forall y)(R(x, y, z))$.

Proof. Induction holds in $\mathfrak{A}$.

LEMMA 14. *For no* $\prod_2$ *or* $\sum_2$ $\mathscr{L}(K_0)$-*formula* $\varphi(z, w)$ *does* $\mathfrak{A} \models L(z) = w \leftrightarrow \varphi(z, w)$.

Proof. From the completeness of $\{z: \mathfrak{A} \models L(z) = 0\}$, and $\mathfrak{A} \models L(z) = 0 \vee L(z) = 1$.

THEOREM 4. *There are disjoint finite relational types* X, Y, *a quantifier free* X-*formula* φ, *and a quantifier free* $X \cup Y$-*formula* ψ, *such that the hypotheses of the weak* (X, Y)-*definability theorem hold, yet the conclusion does not hold for any* $\sum_2$ *or* $\prod_2$ *formulas.*

Theories that consist of only equations $s = t$, for terms s, t, are of special interest to universal algebraists. We now show that there is no bound on the number of alterations of quantifiers needed in the conclusion of the (X, Y)-definability theorem, even if the ψ satisfying its hypotheses is a finite conjunction of equations.

As before, fix $\langle x, y \rangle$, $(z)_0$, $(z)_1$ to be recursive functions such that $\langle\ \rangle$ is one–one, and $\langle (x_0)_0, (x)_1 \rangle = x$. For $A \subset \omega$, let $A^{(n)}$ be the nth Turing jump of A, $A^{(0)} = A$.

Let $F(a_0, \ldots, a_n)$, $2 \leqslant n$, be a function on ω. We let $F^+(a_0, \ldots, a_{n-2})$ be the function given by $F(a_0, \ldots, a_{n-2}, F^+(a_0, \ldots, a_{n-2}), a_n) = F(a_0, \ldots, a_{n-2}, a_n, F^+(a_0, \ldots, a_{n-2})) = a_n$. Such a function F^+ may not exist, in which case F^+ is considered undefined; if F^+ exists then it is obviously unique. Take $F^*(a_0, \ldots, a_{n-2}) = (F^+(a_0, \ldots, a_{n-2}))_0$.

Let $F(a_0, \ldots, a_{2n})$, $0 \leqslant n$, be a function on ω. For $k \leqslant n$, let F^{k*} be defined by $F^{0*} = F^*, F^{(k+1)*} = (F^{k*})^*$. Note that (F^{n*}) is unary, if it exists.

LEMMA 15. *Let* $A \subset \omega$, $1 \leqslant k$, *and let* f *be a* k-*ary function* $\prod_1^0$ *in* A. *Then there is a* $k+2$-*ary function* $F \leqslant_T A$ *such that* $F^+ = f$.

Proof. Define F by $F(a_0, \ldots, a_k, a_k) = a_k$; $F(a_0, \ldots, a_k, a_{k+1}) = a_k$ if a counterexample to $f(a_0, \ldots, a_{k-1}) = a_k$ appears before any counterexample to $f(a_0, \ldots, a_{k-1}) = a_{k+1}$; $F(a_0, \ldots, a_k, a_{k+1}) = a_{k+1}$ if a counterexample to $f(a_0, \ldots, a_{k-1}) = a_{k+1}$ appears before any counterexample to $f(a_0, \ldots, a_{k-1}) = a_k$.

LEMMA 16. *Let* $A \subset \omega$, $1 \leqslant k$, *and let* G *be a* k-*ary function recursive in* $A^{(1)}$. *Then there is a* $k+2$-*ary function* $F \leqslant_T A$ *such that* $F^* = G$.

Proof. This follows from Lemma 15 and the fact every function recursive in $A^{(1)}$ is the result of composing some function $\prod_1^0$ in A with $(\ )_0$.

LEMMA 17. *Let $A \subset \omega$, $0 \leqslant n$, and let h be a unary function recursive in $A^{(n)}$. Then there is a $2n+1$-ary function $F \leqslant_T A$ such that $F^{n*} = h$.*

Proof. By induction on n. The basis case $n = 0$ is trivial. Suppose this is true for n. Let h be a unary function recursive in $A^{(n+1)}$. Then $h \leqslant_T (A^{(1)})^{(n)}$. By induction hypothesis, let G be a $2n+1$-ary function, $G \leqslant_T A^{(1)}$, with $G^{n*} = h$. By Lemma 16, let F be a $2n+3$-ary function with $F \leqslant_T A$ and $F^* = G$. Then $F^{n+1*} = (F^*)^{n*} = G^{n*} = h$, and we are done.

Let X_0 be the relational type $\{\langle\ \rangle, (\ )_0, (\ )_1\}$. For $0 < n$, we let X_n be the relational type consisting of X_0 together with the $(2n+1-2k)$-ary F_k^n, G_k^n, for $0 \leqslant k \leqslant n$.

Let S consist of the equations $F_0^n(x_0, \dots, x_{2n}) = G_0^n(x_0, \dots, x_{2n})$, $G_k^n(x_0, \dots, x_{2n-2k}) = (F_k^n(x_0, \dots, x_{2n-2k}))_0$, $F_{k+1}(x_0, \dots, x_{2n-2k}, G_k^n(x_0, \dots, x_{2n-2k}), x_{2n-2k+2}) = F_{k+1}^n(x_0, \dots, x_{2n-2k}, x_{2n-2k+2}, G_k^n(x_0, \dots, x_{2n-2k})) = x_{2n-2k+2}$. Let S_n, $0 < n$, be the portion of S involving only symbols from X_n.

LEMMA 18. *Let $0 < n$. Then for all X_n-structures $\mathfrak{A}, \mathfrak{B} \models S_n$, if $\mathfrak{A}_{X_0 \cup (F_0^n)} = \mathfrak{B}_{X_0 \cup (F_0^n)}$ then $\mathfrak{A} = \mathfrak{B}$. In addition, there is no $\sum_n$ or $\prod_n$ $X_0 \cup \{F_0^n\}$-formula $C(x, y)$ such that $S_n \models (\forall x)(\forall y)(G_n^n(x) = y \leftrightarrow C(x, y))$.*

Proof. The first part is obvious. For the second, let $A \subset \omega$ be arbitrary, and choose h to be a unary function recursive in $A^{(n)}$, but not $\sum_n^0$ or $\prod_n^0$ in A. By Lemma 17, let F be a $2n+1$-ary function, $F \leqslant_T A$, with $F^{n*} = h$. Let $\mathfrak{A}$ be the unique X_n-structure with domain ω, in which $\langle\ \rangle, (\ )_0, (\ )_1$ are interpreted as $\langle\ \rangle, (\ )_0, (\ )_1$, and F_0^n is interpreted as F. Then G_n^n is interpreted as $F^{n*} = h$. Since $\langle\ \rangle, (\ )_0, (\ )_1, F_0^n$ are all interpreted to be recursive in A, and G_n^n is interpreted to be neither $\sum_n^0$ nor $\prod_n^0$ in A, clearly no such $\sum_n$ or $\prod_n$ formula C can exist.

Now let T_0 consist of the equations $\langle(x)_0, (x)_1\rangle = x$, $(\langle y, z\rangle)_0 = y$, $(\langle y, z\rangle)_1 = z$. We introduce the abbreviations $P_2(x, y) = \langle x, y\rangle$, $P_{n+1}(x_1, \dots, x_{n+1}) = P_2(x_1, P_n(x_2, \dots, x_{n+1}))$, $U_1(x) = (x)_0$, $V_1(x) = (x)_1$, $U_{m+1}(x) = U_1(V_m(x))$, $V_{m+1}(x) = V_1(V_m(x))$, where $1 \leqslant m$, $2 \leqslant n$.

LEMMA 19. *For $2 \leqslant n$, $1 \leqslant m < n$, we have $T_0 \models P_n(U_1(x), \dots, U_{n-1}(x), V_n(x)) = x$, $T_0 \models U_m(P_n(x_1, \dots, x_n)) = x_m$, $T_0 \models V_n(P_n(x_1, \dots, x_n)) = x_n$.*

Proof. By inspection.

The theories S_n involve different relational types, and numbers of equations. We can use Lemma 19 to collapse several variables into one, several function symbols into one, and several equations into one, to obtain the following lemma.

LEMMA 20. *For $0 < n$, we can associate an equation $s = t$ involving only symbols from X_0 or the two unary function symbols F, G, such that the following holds. For all $X_0 \cup \{F\}$-structures $\mathcal{A}$, $\mathcal{B} \models T_0 \cup \{s = t\}$, if $\mathcal{A}_{X_0 \cup \{F\}} = \mathcal{B}_{X_0 \cup \{F\}}$ then $\mathcal{A} = \mathcal{B}$. In addition, there is no $\sum_n$ or $\prod_n$ $X_0 \cup \{F\}$-formula $C(x, y)$ such that $T_0 \cup \{s = t\} \models (\forall x)(\forall y)(G(x) = y \leftrightarrow C(x, y))$.*

THEOREM 5. *There is a unary function symbol G and a finite relational type X consisting of one binary function symbol and three unary function symbols such that the following holds. For each $0 < n$ we can effectively associate a conjunction ψ of four equations, obeying the hypotheses of the $(X, X \cup \{G\})$-definability theorem, such that the conclusion does not hold for any $\sum_n$ formulas.*

Proof. Obvious from Lemma 20 and the fact that T_0 has three axioms.

Let ψ satisfy the hypotheses of the (X, Y)-definability theorem. Then $\{\mathcal{B}_X : \mathcal{B} \models \psi$ and $\mathcal{B}$ is an $X \cup Y$-structure$\}$ is an elementary class. This is an immediate consequence of the (X, Y)-definability theorem. In fact, if there are $\sum_n$ formulas obeying the conclusion of the (X, Y)-definability theorem for ψ, and ψ is a $\sum_m$-formula, then $\{\mathcal{B}_X : \mathcal{B} \models \psi$ and $\mathcal{B}$ is an $X \cup Y$-structure$\}$ is axiomatized by a $\sum_{n+m}$ formula.

Let ψ_X be the set of all consequences of ψ that are X-formulas. Obviously ψ_X is the set of all formulas that hold in all elements of the *EC* class $\{\mathcal{B}_X : \mathcal{B} \models \psi$ and $\mathcal{B}$ is an $X \cup Y$-structure$\}$. From this, we immediately obtain the following.

PROPOSITION. *Let ψ obey the hypotheses of the (X, Y)-definability theorem. If ψ is $\sum_m$, and there are $\sum_n$ formulas obeying its conclusion, then ψ_X is axiomatized by a $\sum_{n+m}$ formula.*

We therefore see that Theorem 6 below is a strengthening of Corollary 1.

LEMMA 21. *There is a $\mathcal{L}(K(0))$-formula α such that the following is* *true. If $\varphi_e(0)\uparrow$ in exactly k steps, then every instance of induction for $\sum_k^0$ formulas is a consequence of $K(e)+\alpha$, and $K(e)+\alpha$ has an ω-model.*

Proof. Take α to be a suitable instance of induction for $\mathscr{L}(K(0))$-formulas.

LEMMA 22. *There is a number q such that the following is true. If $\varphi_e(0)\uparrow$ in exactly $k+q$ steps, then every instance of induction for $\sum_k$ $\mathscr{L}(T_2)$-formulas is a consequence of $K(e)+\alpha$, and $K(e)$ has an ω-model.*

THEOREM 6. *There are disjoint finite relational types X, Y, such that the following holds. There is no recursive function ρ on ω, such that for all quantifier free X-formulae ψ obeying the hypotheses of the (X, Y)-definability theorem, there is a $\sum_{\rho(\#(\psi))}$ axiomatization of ψ_X.*

Proof. This follows from Lemma 22, together with the fact that no consistent set of $\sum_k$ $\mathscr{L}(T_2)$-formulas can have all instances of induction for $\sum_{k+1}$ $\mathscr{L}(T_2)$-formulas as consequences.

Theorem 5 is likewise strengthened by Theorem 7 below.

LEMMA 23. *Let $A\subset\omega$, $0\leqslant n$, and let h be a 3-ary function recursive in $A^{(n)}$. Then there is a $2n+3$-ary function $F\leqslant_T A$ such that $F^{n*}=h$.*

Proof. Same as for Lemma 21.

LEMMA 24. *For each $0\leqslant n$ there is a recursive function H_n such that the following holds. If e is the gödel number of a 3-ary function h recursive in $\omega^{(n)}$, then $H_n(e)$ is the gödel number of a $2n+3$-ary recursive function F such that $F^{n*}=h$.*

LEMMA 25. *For each $0\leqslant n$ there is a recursive function I_n such that the following holds. For all e, $e\notin\omega^{(n+1)}$ iff $I_n(e)$ is the gödel number of a 3-ary function h recursive in $\omega^{(n)}$ such that h^+ exists. For all e, $I_n(e)$ is the gödel number of a 3-ary function recursive in $\omega^{(n)}$.*

Proof. Take $I_n(e)$ to be the gödel number of the 3-ary function h given by $h(x,y,z)=z+1$ if $\varphi_X^{\omega^{(n)}}(x)\uparrow$ in exactly z steps, and $y=0$; 0 otherwise.

LEMMA 26. *For each $0\leqslant n$, there is a recursive function J_n such that the following holds. For all e, $e\notin\omega^{(n+1)}$ iff $J_n(e)$ is the gödel number of a $2n+3$-ary recursive function F such that F^{n+1*} exists.*

LEMMA 27. *If* $2 < n$, *then* $\{e$: *e is the gödel number of a recursive function F such that* F^{n*} *exists*$\}$ *is not* $\sum_n^0$.

THEOREM 7. *There is a unary function symbol G, and a finite relational type X consisting of one binary function symbol and three unary function symbols such that the following holds. For each* $0 < n$ *we can effectively associate a conjunction* ψ *of four equations, obeying the hypotheses of the* $(X, X \cup \{G\})$-*definability theorem, such that there is no* $\sum_n$ *axiomatization of* ψ_X.

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